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Math 12 Honors: HW Section 3.4 Vieta's formula:

1. Given the roots, find the value of the coefficients without expanding:

a) $x^2+bx+c=(x-5)(x+7)$

Roots: 5, -7

$-b=r_1+r_2=5-7=-2 \Rightarrow \boxed{b=2}$

$c=r_1r_2=(5)(-7)=-35 \Rightarrow \boxed{c=-35}$

b) $ax^2+bx+c=(2x+5)(3x-7)$

$a=2\cdot3=6$

$\frac{-b}{a}=\frac{-b}{6}=-\frac{5}{2}+\frac{7}{3} \Rightarrow \boxed{b=1}$

$\frac{c}{a}=\frac{c}{6}=\left(-\frac{5}{2}\right)\left(\frac{7}{3}\right) \Rightarrow \boxed{c=-35}$

c) $x^3+bx^2+cx+d=(x-2)(x+3)(x-4)$

$r_1r_2r_3=2,-3,4$

$-b=2-3+4 \Rightarrow \boxed{b=-3}$

$c=(2)(-3)+(2)(4)+(-3)(4)=-10 \Rightarrow \boxed{c=-10}$

$-6 \quad 8 \quad -12$

$d=(2)(-3)(4)=-24 \Rightarrow \boxed{d=-24}$

d) $ax^3+bx^2+cx+d=(x-3)^2(2x+1)$

$r_1,r_2,r_3=3,3,-\frac{1}{2}$

$a=2$

$-\frac{b}{2}=3+3-\frac{1}{2} \Rightarrow \boxed{b=-11}$

$\frac{c}{2}=3\cdot3+3\cdot\left(-\frac{1}{2}\right)+3\cdot\left(-\frac{1}{2}\right) \Rightarrow \boxed{c=12}$

$-\frac{d}{2}=3\cdot3\cdot\left(-\frac{1}{2}\right) \Rightarrow \boxed{d=9}$

e) $x^4+bx^3+cx^2+dx+e=(x-3)(x+3)(x-5)(x+5)$

$r_1,r_2,r_3,r_4=3,-3,5,-5$

$-b=3-3+5-5 \Rightarrow \boxed{b=0}$

$c=-9+\cancel{15}-\cancel{15}-\cancel{15}+\cancel{15}-25 \Rightarrow \boxed{c=-34}$

$d=-\cancel{45}+\cancel{45}-\cancel{75}+\cancel{75} \Rightarrow \boxed{d=0}$

$e=3\cdot(-3)\cdot5\cdot(-5)=225 \Rightarrow \boxed{e=225}$

f) $ax^4+bx^3+cx^2+dx+e=(2x+3)^2(x-4)(x+3)$

$a=4$

$r_1,r_2,r_3,r_4=-\frac{3}{2},-\frac{3}{2},4,-3$

$-\frac{b}{4}=-\frac{3}{2}-\frac{3}{2}+4-3 \Rightarrow \boxed{b=8}$

$\frac{c}{4}=\frac{9}{4}-6+\frac{9}{2}-6+\frac{9}{2}-12 \Rightarrow \boxed{c=-51}$

$-\frac{d}{4}=9-\frac{27}{4}+18+18 \Rightarrow \boxed{d=-153}$

$\frac{e}{4}=\left(-\frac{3}{\cancel{2}}\right)\left(-\frac{3}{\cancel{2}}\right)\cdot\cancel{4}\cdot(-3) \Rightarrow \boxed{e=-108}$

2. What is the sum of the squares of the roots of $4x^2+6x+2=0$?

Roots: r_1, r_2

$r_1^2+r_2^2=?$

$-\frac{b}{a}=r_1+r_2 \Rightarrow -\frac{6}{4}=r_1+r_2 \Rightarrow \frac{9}{4}=r_1^2+2r_1r_2+r_2^2$

$\frac{c}{a}=r_1r_2 \Rightarrow \frac{2}{4}=r_1r_2 \Rightarrow 1=2r_1r_2$

subtract $r_1^2+r_2^2=\frac{9}{4}-1=\boxed{\frac{5}{4}}$

3. If three roots of $x^4+Ax^2+Bx+C=0$ are -1, 2, and 3, then what is the value of $2C-AB$?

$f(x)=(x+1)(x-2)(x-3)(x-r_4)$

$=x^4+0x^3+Ax^2+Bx+C=0$

$-0=-1+2+3+r_4 \Rightarrow r_4=-4$

$A=-2-3+\cancel{4}+6-\cancel{8}-\cancel{12}=-15$

$-B=-6+8+12+24=-10$

$C=(-1)(2)(3)(-4)=24$

$2C-AB=12r_4+(r_4+4)(1+4r_4)$

$\boxed{2C-AB=48-(-15)(+10)=198}$

4. Given that the roots of $f(x) = x^3 - 4x^2 + 15x - 7$ are "a", "b", and "c", find the value of

$i) a+b+c$ $\qquad$ $ii) ab+bc+ac$ $\qquad$ $iii) a \times b \times c$

$ax^3 + bx^2 + cx + d$

$-\frac{b}{a} = r_1 + r_2 + r_3$

$+\frac{c}{a} = r_1r_2 + r_1r_3 + r_2r_3$

$-\frac{d}{a} = r_1r_2r_3$

$= \boxed{4}$ $\qquad$ $= \boxed{15}$ $\qquad$ $= \boxed{7}$

5. Given that ""a", "b", "c" and "d" are the roots of $f(x) = 3x^4 + 2x^3 - 7x^2 + 9x - 10$, find the value of $abc + abd + acd + bcd$

$-\frac{9}{3} = abc + abd + acd + bcd$

$\boxed{-3}$

6. Given that "a", "b", and "c" are the roots of $f(x) = 3x^3 - 4x^2 + 5x + 7$ find the value of the following:

$i)\ a+b+c$ $\qquad$ $ii) a^2 + b^2 + c^2$ $\qquad$ $iii)\ \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

$\boxed{\frac{4}{3}}$

$\left(\frac{4}{3}\right)^2 - 2\left(\frac{5}{3}\right) = \boxed{-\frac{14}{9}}$

$(2ab + 2ac + 2bc)$

$(a^2 + b^2 + c^2 + 2ab + 2ac + 2bc)$

$\frac{ab + ac + bc}{abc} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

$\frac{\left(\frac{5}{3}\right)}{-\frac{7}{3}} = \boxed{-\frac{5}{7}}$

7. What is the sum of the reciprocals of the 5 solutions to: $5x^5 + 4x^4 - 3x^3 + 2x^2 + x - 1 = 2000$?

Roots: r_1, r_2, r_3, r_4, r_5

$5x^5 + 4x^4 - 3x^3 + 2x^2 + x - 2001 = 0$

$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} + \frac{1}{r_5} = \frac{\text{groups of 4 roots}}{r_1r_2r_3r_4} = \frac{\frac{1}{5}}{\frac{-2001}{5}} = \boxed{\frac{1}{2001}}$

$\frac{e}{a} = \frac{1}{5}$ $\qquad$ $\frac{-f}{a} = \frac{2001}{5}$

8. What is the arithmetic mean of the four solutions of $x^4 - 28x^3 + 265x^2 - 1006x + 1560 = 0$?

Roots: r_1, r_2, r_3, r_4

Sum of roots $= -\frac{b}{a} = 28$ $\qquad$ Mean $= \frac{28}{4} = \boxed{7}$

9. The polynomial equation $x^3 - 6x^2 + 5x - 1 = 0$ has three real roots: a, b, and c. Determine the value of $a^5 + b^5 + c^5$.

See assached paper for solution

10. $f(x)=x^4+6x^3+ax^2-54x+c$ has four real roots: r_1, r_2, r_3, *and* r_4 such that $r_1+r_2=0$ and $r_4-r_3=4$. Find the values of coefficients "a" and "c". (AoPS)

$-6 = \cancel{r_1+r_2}+r_3+r_4 = r_3+r_4$; $r_3+r_4=-6$, $r_4-r_3=4$ $\Rightarrow 2r_4=-2 \Rightarrow \boxed{r_4=-1,\ r_3=-5}$

$54 = r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4 = -5r_1r_2 - r_1r_2+5r_1+5r_2 = -6r_1r_2+\cancel{5r_1+5r_2}$

$-6r_1r_2=54$

$r_1r_2=-9$

$\boxed{r_2=3,\ r_1=-3}$

$c=(-1)(-5)(3)(-3)=-45$

$a = r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4 = \boxed{-4}$

$\boxed{(a,c)=(-4,-45)}$

11. Suppose $f(x)=x^3-5x^2+12x-19$ with roots "a", "b", and "c", what is the value of $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$?

$\frac{a+b+c}{abc} = \frac{1}{bc}+\frac{1}{ac}+\frac{1}{ab}$

$a+b+c=5$

$abc=19$

$\frac{1}{ac}+\frac{1}{ab}+\frac{1}{bc} = \boxed{\frac{5}{19}}$

12. Suppose $r_1, r_2, r_3, \ldots\ldots r_{20}$ are the roots of $f(x)=x^{20}-19x+2$. What is the value of $r_1^{20}+r_2^{20}+r_3^{20}+\ldots+r_{20}^{20}$?

$= -\frac{B}{A} = 19$

$f(r_1) = r_1^{20}-19r_1+2=0$

$f(r_2) = r_2^{20}-19r_2+2=0$

$\vdots$

$f(r_{20}) = r_{20}^{20}-19r_{20}+2=0$

$r_1^{20}+r_2^{20}+\dots+r_{20}^{20} - 19(r_1+r_2+\dots+r_{20}) + 20(2) = 0$

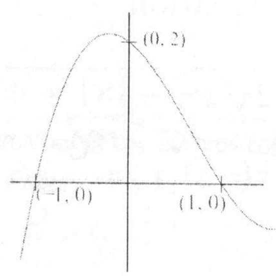
$r_1^{20}+r_2^{20}+\dots+r_{20}^{20} = 19^2-40 = \boxed{321}$

13. What is the sum of all the coefficients (including the constant term) in the polynomial expansion of

$f(x) = (x-1)(x-2)(x-3)(x-4)\times\ldots\ldots\times(x-1996)(x-1997)$

$f(1) = a+b+c+d+\dots+\text{constant} = 0(-1)(-2)\dots(-1996) = \boxed{0}$

14. Part of the graph of $y=f(x)=ax^3+bx^2+cx+d$ is shown. What is the value of "b"?



$d=2$

$r_1=-1$

$r_2=1$

$r_3=\frac{2}{a}$

$\frac{-d}{a} = r_1r_2r_3 = -r_3$

$\frac{2}{a} = r_3 \Rightarrow r_3 = \frac{2}{a}$

$-\frac{b}{a} = \cancel{r_1+r_2}+r_3 \Rightarrow -\frac{b}{a} = \frac{2}{a} \Rightarrow \boxed{b=-2}$

15. Challenge: Let "a", "b", and "c" be the roots of $x^3+3x^2-24x+1=0$. Suppose all three roots are real, show that $\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}=0$ (AoPS) 1 3 3 1

1

$x^3+3x^2-24x+1=0$

$x^3+3x^2+3x+1=27x$

$(x+1)^3=27x$

$x+1=3\sqrt[3]{x} \Rightarrow a+1=3\sqrt[3]{a}, \quad b+1=3\sqrt[3]{b}, \quad c+1=3\sqrt[3]{c}$

2

3

Sum: $\underbrace{a+b+c}_{=-3 \text{ by vieta}}+3=3(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c})=$

$\boxed{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}=0}$

(2003 AMC 10A #18) What is the sum of the reciprocals of the roots of the equation

$$\frac{2003}{2004}x+1+\frac{1}{x}=0?$$

$$\frac{2003x^2}{2004}+x+1=0$$

Sum of reciprocal $=\dfrac{r_1r_2+r_1r_3+r_2r_3}{r_1r_2r_3}=\dfrac{\frac{-2004}{2003}}{\frac{2004}{2003}}=\boxed{-1}$

(1991 MAΘ) Let r, s, t be the roots of $x^3-6x^2+5x-7=0$. Find

$$\frac{1}{r^2}+\frac{1}{s^2}+\frac{1}{t^2}.$$

$rs+rt+st=5$

$r^2s^2+r^2t^2+s^2t^2+2rs^2t+2r^2st+2rst^2=25$ divide $\rightarrow \dfrac{25}{49}=\dfrac{1}{r^2}+\dfrac{1}{s^2}+\dfrac{1}{t^2}+\dfrac{2}{st}+\dfrac{2}{sr}+\dfrac{2}{rt}$

$rst=7$

$r^2s^2t^2=49$

$r+s+t=6 \Rightarrow \dfrac{6}{7}=\dfrac{1}{st}+\dfrac{1}{sr}+\dfrac{1}{rt} \Rightarrow \dfrac{12}{7}=\dfrac{2}{st}+\dfrac{2}{sr}+\dfrac{2}{rt}$

subtract $\rightarrow \dfrac{1}{r^2}+\dfrac{1}{s^2}+\dfrac{1}{t^2}=\dfrac{25}{49}-\dfrac{12}{7}=\boxed{\dfrac{-59}{49}}$

(1983 AIME) What is the product of the real roots of the equation

$$x^2+18x+30=2\sqrt{x^2+18x+45}?$$

add 15 to both sides

$x^2+18x+45-2\sqrt{x^2+18x+45}=15$

Let $A=\sqrt{x^2+18x+45}$.

$A^2-2A-15=0$

$(A+3)(A-5)=0$

$A_1=-3$

$A_2=5$

$\cancel{\sqrt{x^2+18x+45}=(-3)^2}$ rej. not real

$x^2+18x+45=25$

$x^2+18x+20=0$

$\boxed{\text{by vieta, the product of the roots is equal to } 20}$

Given $x^3-6x^2+5x-1=0$, find $a^5+b^5+c^5$ Question #9 3.4 Mahyar Pregash

$a+b+c=6$

$ab+ac+bc=5$

$abc=1$

$a+b+c=\underline{6}$

$a^2+b^2+c^2=(a+b+c)^2-2(ab+ac+bc)=36-10=\underline{26}$

$a^3+b^3+c^3=(a^2+b^2+c^2)(a+b+c)-\left[(ab+ac+bc)(a+b+c)-3(abc)\right]=26\cdot6-(30-3)=129$

$a^4+b^4+c^4=(a^3+b^3+c^3)(a+b+c)-\left[(a^2+b^2+c^2)(ab+ac+bc)-abc(a+b+c)\right]=650$

$a^5+b^5+c^5=(a^4+b^4+c^4)(a+b+c)-\left[(a^3+b^3+c^3)(ab+ac+bc)-abc(a^2+b^2+c^2)\right]=\boxed{2631}$